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2+1 GRAVITY, CHAOS AND TIME MACHINES

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Abstract

2+1 gravity for spacetimes with topology $\mathbf{R} \times \mathbf{T}^2$ has been much studied. We add a description of how to extend these spacetimes across a Cauchy horizon into a region where the torus becomes Lorentzian. The result is a one parameter family of tori given by a geodesic in the "Teichmüller space" of Lorentzian tori. We describe this in detail. We also point out that if the modular group is regarded as part of the gauge group then these spacetimes offer a nice toy model for the dynamics of Bianchi IX models; in the region where the tori are spacelike the dynamics is described exactly by a hyperbolic billiard. On the other hand the modular group acts ergodically on the Teichmüller space of Lorentzian tori.

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1. INTRODUCTION.

The subject of 2+1 dimensional gravity looks *a priori* unpromising since—in the absence of matter—all spacetimes have constant curvature. Nevertheless it has been the subject of many investigations over the past twenty years or so. Indeed it is now widely recognized that it provides (when handled with taste!) surprisingly illuminating toy models of general relativity. Most of these investigations center on quantum gravity [1], often from a Hamiltonian point of view, and as a result the spacetime properties of the models are receiving somewhat less attention than we think that they deserve. Here we intend to present some properties of 2+1 spacetimes with topology $\mathbf{R} \times \mathbf{T}^2$, regarded as quotients of Minkowski space. This perspective enables us to discuss what goes on in that region of spacetime where the torus becomes Lorentzian and closed timelike curves appear; existing treatments [2] [3] typically use the Hamiltonian ADM formalism and therefore do not go across the Cauchy horizon that bounds this region. The motivation for doing this is partly just curiosity, but partly a feeling that there is structure there which may well illuminate some features occurring in 3+1 dimensions too—even if it will manifest itself in a different way in the latter case. Be that as it may a nice picture emerges; we can regard the entire spacetime as a geodesic in the Teichmüller space of tori. This space is the familiar upper half plane in the Riemannian case, and it is 1+1 dimensional de Sitter space in the Lorentzian case.

The second point that we wish to bring up is that if the modular group is regarded as part of the gauge group then these spacetimes offer a nice toy model for the chaotic behaviour of Bianchi cosmologies. The dynamics of the latter has attracted attention for quite some time and many of its aspects are by now well understood. (There are many references, old [4] [5], new [6] and very new [7].) In particular it is well known that the behaviour of Bianchi IX models close to the singularity can be approximated by a hyperbolic billiard, which is an archetypical chaotic system. In the literature the situation is often described by saying that chaotic behaviour appears when curvature becomes strong, although the precise meaning of the word "chaotic" here is a subject of some controversy. It is therefore of some interest that this kind of chaotic behaviour appears in 2+1 gravity with zero curvature, as a kind of global effect. A simplifying feature is that in our case the hyperbolic billiard captures the dynamics exactly.

As additional motivation we note that both the points we raise are impor-

tant for quantization. They also appear to be of interest in string theory—see ref. [8], but beware of some misunderstandings in that reference.

The organization of the paper is as follows: In section 2 we construct our spacetimes by taking quotients of a region of 2+1 dimensional Minkowski space with the appropriate discrete isometry groups. This construction is well known [9] [10]. In section 3 we describe these spacetimes as a geodesic in a Teichmüller space; this is a new result as far as the region with closed timelike curves is concerned. Since the Teichmüller space of Lorentzian tori has caused some puzzlement in the past [11] we describe it in detail. In section 4 we describe the dynamics which results when taking the quotient of Teichmüller space with the modular group, and stress the analogy to mixmaster cosmology. We focus on the spectrum of closed geodesics since they are the skeleton on which chaos is built; actually a closed geodesic corresponds to a self-similar rather than a periodic spacetime. Our account is intended to be pedagogical (and to be helpful in section 5); all the hard results are well known to mathematicians [12] [13] and to workers in quantum chaos [14]. In section 5 we discuss the action of the modular group on the Teichmüller space of Lorentzian tori. We show that it is ergodic. (In a general setting involving discrete groups acting on coset spaces formed from non-compact groups such phenomena are known to mathematicians, but our pedestrian treatment is original as far as we know.) In section 6 we sketch how our method works for locally de Sitter spacetimes [15], and comment on the higher genus case. Our conclusions are in section 7.

2. OUR SPACETIMES.

Let \mathbf{M} be a region of 2+1 dimensional Minkowski space and Γ a free discrete isometry group acting in a properly discontinuous way on this region. We want to choose Γ so that the quotient space \mathbf{M}/Γ has the topology of a torus cross the real line. For a simply connected M the quotient space has Γ as its fundamental group. Therefore Γ must be a free discrete group with two commuting generators. We also insist that the quotient space should contain a complete spacelike surface that is not crossed by any closed timelike (or null) curve. The solution to this problem is described, e.g., by Louko and Marolf [10]. As generators of the discrete group we choose $g_1 = e^{\xi_1}$ and $g_2 = e^{\xi_2}$, that is to say exponentials of the two linearly independent commuting Killing vectors

$$\xi_1 = \alpha J_{xt} + \beta P_y \quad \xi_2 = \gamma J_{xt} + \delta P_y ; \quad \alpha\delta - \beta\gamma > 0. \quad (1)$$

Here J_{xt} is a Lorentz boost, P_y is a translation and α, β, γ and δ are real numbers. This is the most general solution, except for the obvious static case that we ignore. The group Γ will contain all group elements of the form e^ξ , where

$$\xi = (n_1\alpha - n_2\gamma)J_{xt} + (n_1\beta - n_2\delta)P_y , \quad n_1, n_2 \in \mathbf{Z} . \quad (2)$$

Here n_1 and n_2 are arbitrary integers. We observe that Γ will contain pure boosts if and only if β/δ is rational, and pure translations if and only if α/γ is rational. Note also that in any case the action of Γ on the line $x = t = 0$ is problematic; if a pure boost is present it has a line of fixed points there, and if not the action of Γ on this line is ergodic. Hence we see why the covering space M is taken to be a subset of 2+1 dimensional Minkowski space only.

Since the Killing vectors ξ_1 and ξ_2 commute they form surfaces, namely

$$t^2 - x^2 = \tau^2 \equiv -\sigma^2 , \quad (3)$$

where τ^2 is some constant (not necessarily positive; if it is not then σ^2 is positive). These surfaces are left invariant by the group Γ , they foliate Minkowski space, they are intrinsically flat and their mean curvature is constant. They turn into tori when we take the quotient with Γ . From now on we take M to be the union of regions I and II of Minkowski space, as defined in figure 1. This means that our quotient spaces will be geodesically incomplete. If we did not restrict M in this way we would obtain what Louko and Marolf [10] accurately describe as a "modest generalization of Misner space"; as far as we can see there is nothing interesting to say about this that goes beyond Misner's original observations [16] which is why we make the restriction. Since each invariant surface contributes a torus to the quotient space we now see that our spacetimes can be described as a one parameter family of flat tori; spacelike tori coming from region I and labelled by τ and Lorentzian tori coming from region II and labelled by σ . The Cauchy horizon $\tau = \sigma = 0$ contributes a null torus.

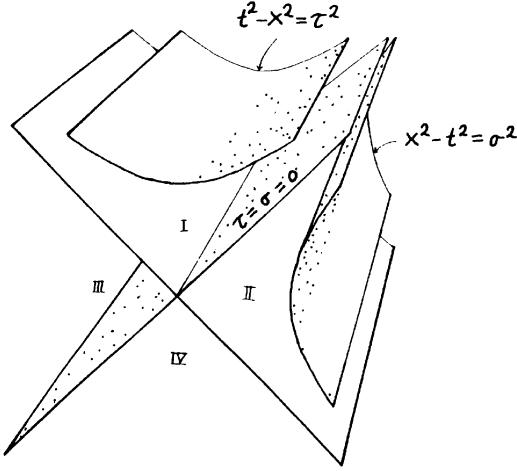


Figure 1: 2+1 dimensional Minkowski space divided into four wedge shaped regions, each of which is foliated by flat surfaces left invariant by Γ . Our covering space consists of regions I and II and our quotient space becomes a one parameter family of tori.

3. A TRIP THROUGH TEICHMÜLLER SPACE.

Our task now is to describe the one parameter family of flat tori that constitutes a spacetime of the kind that we defined in section 2. We use the notation that $\xi_\alpha \xi^\alpha \equiv ||\xi||^2 \equiv \pm |\xi|^2$, where the sign depends on whether the vector is timelike or spacelike and $|\xi|$ is non-negative by definition. Let us first sketch what goes on in the region without closed timelike curves (where a Hamiltonian description is available [2] [3]). At fixed τ the tori are built from parallelograms spanned by the generators ξ_1 and ξ_2 . The angle between them is given by

$$\cos \theta = \frac{\xi_1 \cdot \xi_2}{|\xi_1||\xi_2|} = \frac{\beta\delta + \alpha\gamma\tau^2}{\sqrt{\beta^2 + \alpha^2\tau^2}\sqrt{\delta^2 + \gamma^2\tau^2}}. \quad (4)$$

Therefore their area is a monotonically increasing function:

$$A = |\xi_1||\xi_2| \sin \theta = (\alpha\delta - \beta\gamma)\tau . \quad (5)$$

(The total area of the torus is the area of a parallelogram times a fixed numerical factor that can be chosen at will.) On the other hand the shape of the torus is changing in an interesting way. To describe it we introduce their Teichmüller space:

Definition: Teichmüller space is the moduli space of marked flat tori.

”Marked” means that a particular pair of intersecting closed geodesics on the torus (namely the one that corresponds to our generators ξ_1 and ξ_2) has been singled out for special attention. The definition applies equally well to Riemannian and Lorentzian tori; in the former case it is well known that Teichmüller space can be regarded as the upper half plane, and that it is naturally equipped with the Poincaré metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2) . \quad (6)$$

This is hyperbolic space \mathbf{H}^2 and its isometry group is $PSL(2, \mathbf{R})$. We can assign a position in Teichmüller space to our tori if we first normalize our generators so that ξ_1 has length one and lies along the x -axis. Then the tip of ξ_2 will point at a unique point in the upper half plane, namely

$$(x, y) = \frac{|\xi_2|}{|\xi_1|}(\cos \theta, \sin \theta) = \frac{1}{\beta^2 + \alpha^2\tau^2}(\beta\delta + \alpha\gamma\tau^2, (\alpha\delta - \beta\gamma)\tau) . \quad (7)$$

Note that at this stage we use an auxiliary Euclidean metric on the coordinate plane to assign a point to ξ_2 . We now have a curve parametrized by τ and it is elementary to show that this is a semi-circle meeting the boundary at right angles:

$$\left(x - \frac{\beta\gamma + \alpha\delta}{2\alpha\beta}\right)^2 + y^2 = \left(\frac{\alpha\delta - \beta\gamma}{2\alpha\beta}\right)^2 . \quad (8)$$

This is a geodesic with respect to the natural metric. Hence the statement that the torus evolves along a geodesic in Teichmüller space. It should not be forgotten that it also grows in area. A minor calculation informs us that if we move a distance L along the geodesic, as measured by the Poincaré metric, then the area of the torus grows with a factor e^L . Note that this does not depend on the parameters describing the spacetime, nor does it depend on where we are on the geodesic.

Now what happens when we pass the Cauchy horizon and enter region II? The first observation is that

$$\|\xi_1\|^2 = \beta^2 - \sigma^2\alpha^2 . \quad (9)$$

Hence (unless ξ_1 is a pure translation or a pure boost) ξ_1 is spacelike in a region where $x^2 - t^2 = \sigma^2 < \beta^2/\alpha^2$ and it is timelike when $x^2 - t^2 = \sigma^2 > \beta^2/\alpha^2$. Let us refer to these regions as region IIa and IIb, respectively. To avoid misunderstandings, because the group Γ contains all the elements listed in eq. (2) there are closed timelike geodesics through every point in region II, although the existence of closed null geodesics on the Cauchy horizon depends on whether δ/β is rational or not.

If we now try to mimic the construction of the Teichmüller space of Riemannian tori we run into a problem with the first step, which was to use a rotation to bring the generator ξ_1 into a standard position. We cannot use Lorentz transformations for the same purpose here: The Teichmüller space of Lorentzian tori splits into two components depending on whether ξ_1 is spacelike or timelike. We therefore use a different approach at first. By definition the Teichmüller space is the moduli space of marked flat Lorentzian tori.

Theorem 1: The Teichmüller space of Lorentzian tori has the topology $\mathbf{R} \times \mathbf{S}^1$. It is naturally equipped with the de Sitter metric.

Proof: To each oriented dyad of vectors there corresponds a unique flat marked Lorentzian torus. The set of such dyads is isomorphic to the group $SL(2, \mathbf{R})$. If we perform a Lorentz transformation of the dyad the torus is unchanged. Taking this into account we find a one-to-one correspondence between the Teichmüller space and the coset space $SL(2, \mathbf{R})/SO(1, 1)$. But it is well known that this space has the stated topology. The de Sitter metric

is natural because it is the maximally symmetric metric, and also because it arises if we take the perpendicular distance between the fibers, as measured by the standard metric on $SL(2, \mathbf{R})$.

Although well known the result is not quite trivial. The coset space $SO(2, 1)/SO(1, 1)$ has the topology of the Möbius strip, even though the group manifolds of $SO(2, 1)$ and $SL(2, \mathbf{R})$ have the same topology. Let us give a sketch of the argument: we may, by analogy with the Euler angle parametrization of \mathbf{S}^3 , introduce local coordinates θ, φ, γ on $SL(2, \mathbf{R})$ (aka \mathbf{adS}_3) as

$$\begin{cases} X = \cos \frac{\theta}{2} \sinh \frac{\varphi - \gamma}{2} \\ Y = \sin \frac{\theta}{2} \sinh \frac{\varphi + \gamma}{2} \\ U = \cos \frac{\theta}{2} \cosh \frac{\varphi - \gamma}{2} \\ V = \sin \frac{\theta}{2} \cosh \frac{\varphi + \gamma}{2} \end{cases} \quad (10)$$

The flat metric

$$ds^2 = dX^2 + dY^2 - dU^2 - dV^2 \quad (11)$$

on the embedding space induces the metric

$$ds^2 = \frac{1}{4} (-d\theta^2 + d\varphi^2 + d\gamma^2 - 2d\varphi d\gamma \cos \theta) \quad (12)$$

on $SL(2, \mathbf{R})$. The coordinate γ runs along the flow lines of the Killing field $J_{XU} + J_{YV}$ which generates $SO(1, 1)$ transformations and we want to identify points along these lines. The metric on the resulting space, obtained from the orthogonal distance between the fibers, may be calculated using the threading approach of Boersma and Dray[17]. By identifying the metric in (12) with an Ansatz of the form

$$ds^2 = M^2 (d\gamma - M_i dx^i)^2 + h_{ij} dx^i dx^j, \quad (13)$$

one obtains the metric

$$h = \frac{1}{4} (-d\theta^2 + \sin^2 \theta d\varphi^2) \quad (14)$$

for the quotient space $SL(2, \mathbf{R})/SO(1, 1)$. This is precisely the metric for (part of) **adS**₂ in a reasonably well known coordinate system; anti-de Sitter space and de Sitter space are identical in 1+1 dimensions. It is also possible to do this calculation in global coordinates, at the expense of their not being adapted to the identification Killing field.

As it stands Theorem 1 is not very useful. To see what kind of curve our tori describe we need to know how to assign a point in Teichmüller space to a given marked torus. This understanding will be provided by the proof of Theorem 2, which will wind its way to the end of this section:

Theorem 2: The Teichmüller space of Lorentzian tori has the topology $\mathbf{R} \times \mathbf{S}^1$. The one parameter family of tori that represents a spacetime (defined in section 2) is a timelike geodesic in this space provided that it is equipped with the de Sitter metric.

Proof: Our first step is to introduce coordinates (x, t) . During the construction we use the flat Minkowski metric on this coordinate plane. Again we normalize the vectors so that ξ_1 points at $(1, 0)$. This is always possible provided that $\sigma^2 < \beta^2/\alpha^2$. Since we know the scalar product of the vectors we find that the tip of ξ_2 points at the point

$$(x, t) = \frac{1}{\beta^2 - \sigma^2 \alpha^2} (\beta \delta - \alpha \gamma \sigma^2, -(\alpha \delta - \beta \gamma) \sigma) . \quad (15)$$

We have therefore been able to arrange that this component of Teichmüller space is identical to the lower half plane. It is elementary to show that the points on this curve obey

$$\left(x - \frac{\beta \gamma + \alpha \delta}{2\alpha \beta} \right)^2 - t^2 = \left(\frac{\alpha \delta - \beta \gamma}{2\alpha \beta} \right)^2 . \quad (16)$$

This is a hyperbola with its foci on the x -axis and it is a geodesic with respect to the metric

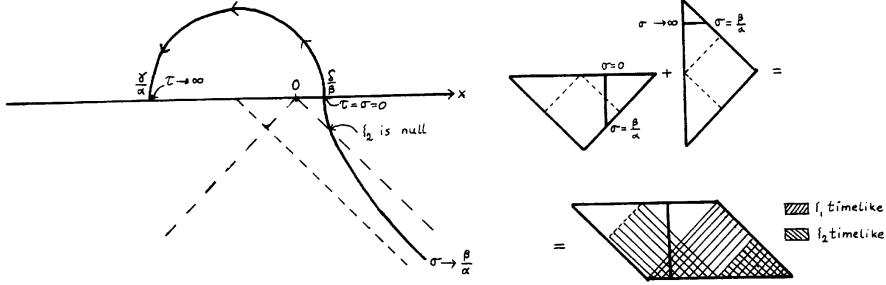


Figure 2: The curve through Teichmüller space. In the upper half plane the torus is Riemannian. In the lower half plane the torus is Lorentzian but the generator ξ_1 is still spacelike. When ξ_1 is timelike we again obtain a half plane. The latter two half planes are conveniently depicted with conformal diagrams; adding them together so that the curve becomes smooth we obtain the conformal diagram of 1+1 dimensional de Sitter space.

$$ds^2 = \frac{1}{t^2}(dx^2 - dt^2) . \quad (17)$$

But this is in fact the de Sitter metric on a coordinate patch that covers "one half" of de Sitter space.

We can now draw a picture of the geodesic in Teichmüller space, where the Teichmüller space of Riemannian tori has been joined to its counterpart for Lorentzian tori across their conformal boundaries. Note that in the Lorentzian part of the picture the geodesic reaches infinite coordinate values at finite parameter values $\sigma^2 = \beta^2/\alpha^2$. This is actually a good thing: We know that the coordinates we are using cover only a part of Teichmüller space. "Infinity" in the picture corresponds to a coordinate singularity that is caused by our assumption that ξ_1 is spacelike.

When ξ_1 is timelike we again introduce an infinite half plane, this time described by the coordinates t' and $x' > 0$, and normalize the vectors so that ξ_1 points at $(t', x') = (1, 0)$. We then find that ξ_2 points at

$$(t', x') = \frac{1}{\sigma^2 \alpha^2 - \beta^2} (\beta \delta - \alpha \gamma \sigma^2, (\alpha \delta - \beta \gamma) \sigma) . \quad (18)$$

These points lie on the hyperbola

$$\left(t' + \frac{\beta\gamma + \alpha\delta}{2\alpha\beta}\right)^2 - x'^2 = \left(\frac{\alpha\delta - \beta\gamma}{2\alpha\beta}\right)^2. \quad (19)$$

This is a geodesic with respect to the metric

$$ds^2 = \frac{1}{x'^2}(dt'^2 - dx'^2). \quad (20)$$

This is again the metric on "one half" of de Sitter space. Since we now think of the conformal boundary as being timelike it may be more natural to think of it as anti-de Sitter space—but in 1+1 dimensions de Sitter space and anti-de Sitter space coincide when we switch the meaning of space and time.

It remains to show that the two components of Teichmüller space can be glued together so that they form a de Sitter space, in such a way that the curve becomes a geodesic globally. For this purpose we observe that both \mathbf{H}^2 (the Teichmüller space of Riemannian tori) and 1+1 dimensional de Sitter space can be isometrically mapped into surfaces in a 2+1 dimensional Minkowski space with the metric

$$ds^2 = dX^2 + dY^2 - dU^2. \quad (21)$$

Explicitly we define an embedding of \mathbf{H}^2 by

$$X = \frac{x}{y} \quad Y + U = \frac{1}{y} \quad Y - U = -\frac{x^2 + y^2}{y}; \quad y > 0. \quad (22)$$

The surface is the upper sheet of the hyperboloid $X^2 + Y^2 - U^2 = -1$ and the induced metric is the one given in eq. (6). The first component of the Teichmüller space of Lorentzian tori is embedded through

$$X = \frac{x}{t} \quad Y + U = \frac{1}{t} \quad Y - U = \frac{t^2 - x^2}{t}; \quad t > 0. \quad (23)$$

The surface is "one half" of the hyperboloid $X^2 + Y^2 - U^2 = 1$ and the induced metric is the one given in eq. (17). The second component is embedded through

$$X = \frac{t'}{x'} \quad Y + U = -\frac{1}{x'} \quad Y - U = \frac{t'^2 - x'^2}{x'} ; \quad x' > 0 . \quad (24)$$

The surface is "the other half" of the hyperboloid $X^2 + Y^2 - U^2 = 1$ and the induced metric is the one given in eq. (20).

A geodesic in \mathbf{H}^2 , and a timelike geodesic in de Sitter space, is uniquely defined as the intersection of a hyperboloid with a timelike plane through the origin in the embedding space. The curve in Teichmüller space is given by eqs. (8), (16) and (19). Therefore, to show that this curve is globally a timelike geodesic in de Sitter space we must find a spacelike vector k_α such that eqs. (16) and (19) are equivalent to $k \cdot X = 0$. An elementary calculation shows that this is the case for the vector

$$(k_X, k_Y, k_U) = (\alpha^2 \delta^2 - \beta^2 \gamma^2, \beta \delta (\alpha^2 + \gamma^2) - \alpha \gamma (\beta^2 + \delta^2), \alpha \gamma (\beta^2 - \delta^2) + \beta \delta (\gamma^2 - \alpha^2)) . \quad (25)$$

Eq. (8) is also reproduced. This completes the proof that the curve is globally described by a timelike geodesic in de Sitter space.

4. THE COGWHEELS OF CHAOS.

In this section we restrict ourselves to region I (where there are no closed timelike curves), so that the evolution can be regarded as time evolution in a configuration space in the standard sense [2] [3]. However, it is a moot point whether the configuration space should be taken to be Teichmüller space or the moduli space of (unmarked) flat tori. The latter space is in fact \mathbf{H}^2/Γ_M , where Γ_M is the modular group $PSL(2, \mathbf{Z})$ acting on the upper half plane through

$$z \rightarrow z' = \frac{az + b}{cz + d} ; \quad ad - bc = 1 \quad (26)$$

where a, b, c and d are integers and $z = x + iy$. (To see that z and z' actually correspond to the same torus, consider a pair of intersecting closed geodesics on the torus and choose them to have the shortest circumference possible. The conformal structure can be characterized by the angle and relative lengths of this pair. A little experimentation shows that these are unaffected by a modular transformation.) The quotient space is the famous modular surface, usually described as the fundamental region of the group which is bounded by $r^2 \equiv x^2 + y^2 = 1$ and $x = \pm 1/2$. It is depicted in fig. 3. Its area is finite and it is a smooth manifold except for two conical singularities occurring at the fixed points of the transformations S and ST , where S and T are the transformations

$$Sz = -\frac{1}{z} \quad Tz = z + 1 . \quad (27)$$

S and T generate the group and obey two relations, viz. $S^2 = 1$ and $(ST)^3 = 1$. Note that the transformation S acts by switching the elements in the oriented dyad that defines the torus.

The question whether the configuration space is \mathbf{H}^2 or \mathbf{H}^2/Γ_M matters for the properties of the model but it is not a question of right or wrong, since we do not intend to compare the model to experiment anyway. Technically the modular group does not belong to the connected component of the gauge group so that both options are open as far as consistency is concerned. For thoughtful comments on this issue we refer to papers by Peldán [18] and Matschull [19]; here we choose the second option because it is an interesting one.

As shown by Artin [12] and Hedlund [20] the geodesic flow on the modular surface is ergodic (indeed they showed this at a time when the proper definition of an ergodic system was yet to be found—with today's definition we can say that the flow has the Bernoulli property, which is the strongest ergodic property around). From this point of view it has been much studied; Series has written a nice review with some entries to the technical literature [13]. Here we focus on one aspect of this flow, namely its closed orbits. We take the point of view that one can define "chaos" in a dynamical system by the requirement that the number of its unstable closed orbits rises exponentially as a function of length. This is not at all unreasonable; in fact this is the feature of chaotic systems that survives the transition to quantum theory (via the Gutzwiller trace formula, which connects the asymptotic properties

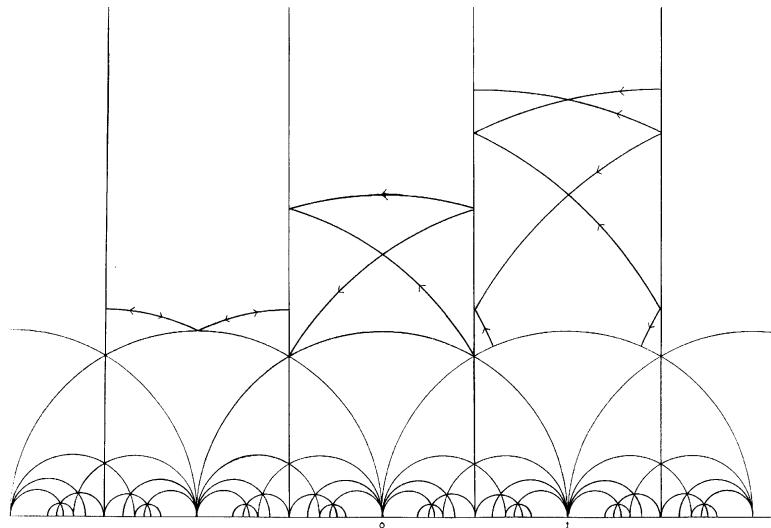


Figure 3: The modular surface is the fundamental region of the modular group, with sides appropriately identified. The picture shows how the upper half plane is tessellated by copies of the fundamental region. In three of the copies we have drawn examples of closed geodesics ($N = 3$ $x_+ = [1]$, $N = 4$ $x_+ = [1, 2]$ and $N = 5$ $x_+ = [1, 3]$ in the notation introduced below).

of the spectrum of closed geodesics to the spectrum of the Laplacian). Since it is a simple matter of counting it is also a feature that survives the transition to diffeomorphism invariant systems—unlike Lyapounov exponents and the like that can be reparametrized away. To avoid confusion, note that—because the area of our tori is growing—a closed geodesic in moduli space actually corresponds to a self-similar rather than a periodic spacetime.

The closed geodesics on the modular surface arise because any hyperbolic Möbius transformation—corresponding to an $SL(2, \mathbf{R})$ matrix whose trace has an absolute value larger than two—has a unique geodesic flowline connecting its pair of fixed points on the real axis. If this Möbius transformation is a modular transformation as well there are points on this geodesic that will be identified with each other, and a closed geodesic results. The distance L between a pair of neighbouring identified points is easily computed. It is given by

$$2 \cosh \frac{L}{2} = N , \quad (28)$$

where $N = |\text{Tr}g|$ and g is the matrix corresponding to the modular transformation, so that $N = a + d$ if the transformation is written as in eq. (26). Note that N can be used to label the conjugacy classes of $SL(2, \mathbf{R})$. This therefore is the length spectrum of the closed geodesics.

It takes more effort to understand how many closed geodesics there are. In group theoretical terms this is the problem to enumerate the conjugacy classes of $PSL(2, \mathbf{Z})$. There are only two conjugacy classes of elliptic elements, corresponding to the two fixed points on the boundary of the fundamental region. The number of conjugacy classes of hyperbolic elements on the other hand is a rapidly growing function of N . It is in fact known (see for instance ref. [14]) that when L is large the number n of closed geodesics with length l not exceeding L grows like

$$n(l \leq L) \sim \frac{e^L}{L} . \quad (29)$$

This settles it: The system is chaotic. It is however an instructive exercise to compute the number of closed geodesics "from below" with pedestrian methods, and this we will now proceed to do.

A geodesic in the upper half plane can be conveniently characterized by two real numbers, its starting point x_+ and its end point x_- on the real axis. To each geodesic we can associate a hyperbolic Möbius transformation whose fixed points are these two points. The geodesic projects to a closed geodesic on the modular surface if and only if this Möbius transformation belongs to the modular group, and there will be a unique such Möbius transformation of smallest trace associated to the closed geodesic (if $x = gx$ then $x = g^n x$; if $n > 1$ the trace of g^n is greater than the trace of g and the corresponding geodesic is traversed several times—here we count only "primitive" closed geodesics). In equations then

$$x_{\pm} = \frac{ax_{\pm} + b}{cx_{\pm} + d} . \quad (30)$$

It follows that x_{\pm} is a quadratic surd, that is to say a solution to a quadratic algebraic equation with integer coefficients whose discriminant is not a perfect square. The two solutions to this equation are

$$x_{\pm} = \frac{1}{2c}(a - d \pm \sqrt{N^2 - 4}) , \quad (31)$$

where $N = a + d$ and we made use of the condition $ad - bc = 1$. Note that the discriminant $D = N^2 - 4 = 4 \sinh^2 \frac{L}{2}$ according to eq. (28). Since the surds occur in pairs the closed geodesics can in fact be labelled by just one real number, say its "source" x_+ .

Next we introduce continued fractions [21]. A real number can be uniquely expressed in the form

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}} \equiv [a_0, a_1, a_2, \dots] , \quad (32)$$

where all the partial quotients a_i are integers and all except possibly a_0 are positive. It is known that x is rational if and only if its continued fraction expansion is finite (i.e. the number of its partial quotients is finite), and it is a quadratic surd if and only if its continued fraction expansion eventually repeats, in which case it is called periodic. The beginning and end of the period is then marked with overdots, so that a quadratic surd of period

length k is of the form $x = [a_0; a_1, \dots, a_{n-1}, \dot{a}_n, \dots, \dot{a}_{n+k-1}]$. This nice characterization of quadratic surds is interesting to us.

One piece of the technology of continued fractions should be mentioned, which is that they give rise to a sequence of approximations of x by rational numbers:

$$[a_0] = \frac{p_0}{q_0} \quad [a_0, a_1] = \frac{p_1}{q_1} \quad [a_0, a_1, a_2] = \frac{p_2}{q_2} \quad (33)$$

and so on. Here p_n and q_n are polynomials in the partial quotients and by induction one can show that

$$p_n = a_n p_{n-1} + p_{n-2} \quad q_n = a_n q_{n-1} + q_{n-2} . \quad (34)$$

Note that p_n and q_n are monotonically increasing functions of n .

We want to count equivalence classes of geodesics under the modular group and therefore we will try to fix one member of each equivalence class. Now the modular group acts on a continued fraction in the following way:

$$\begin{aligned} x = [a_0, a_1, a_2, a_3, \dots] &\rightarrow ST^{-a_0}x = -[a_1, a_2, a_3, \dots] \rightarrow \\ &\rightarrow ST^{a_1}ST^{-a_0}x = [a_2, a_3, \dots] . \end{aligned} \quad (35)$$

It follows that we can remove the partial quotients in pairs. In particular it follows that we can choose x_+ to be a purely periodic continued fraction since we can always remove the initial sequence. Hence without loss of generality

$$x_+ = [\dot{a}_0, \dots, \dot{a}_{k-1}] . \quad (36)$$

If the period length k is even then x_+ is a fixed point of the group element

$$g = ST^{a_{k-1}} \dots ST^{a_1}ST^{-a_0} . \quad (37)$$

In terms of the polynomials introduced above it can be shown that

$$x_+ = gx_+ = \frac{q_{k-2}x_+ - p_{k-2}}{-q_{k-1}x_+ + p_{k-1}} \Rightarrow N = |\text{Tr}g| = p_{k-1} + q_{k-2} . \quad (38)$$

This is a useful fact since it means that N is a monotonically increasing function of the partial quotients. It also means that N will grow when the length of the period in the continued fraction grows, other things being equal.

The fixed point x_+ is in fact the source of the geodesic associated with g . This is so because g removes one period from the continued fraction, so that when g acts on an approximation to x_+ that is a rational number whose continued fraction expansion consists of a finite number of periods then g moves that rational number away from x_+ . If the period length is odd then that g which leaves it fixed and has the smallest value of N is

$$g = ST^{a_{k-1}} \cdot \dots \cdot ST^{a_0} ST^{-a_{k-1}} \cdot \dots \cdot ST^{-a_0} . \quad (39)$$

It is convenient to regard continued fractions of odd period lengths as having even periods of twice the original length. According to a theorem of Galois' the corresponding sink (the other root of the quadratic equation) now obeys

$$Sx_- = -\frac{1}{x_-} = [\dot{a}_{k-1}, \dots, \dot{a}_0] . \quad (40)$$

It is easy to show this since x_- is the source of the group element g^{-1} . The source and sink are now given in reduced form; this means that $x_+ > 1$ and $-1 < x_- < 0$.

Two geodesics in reduced form will give rise to the same closed geodesic on the modular surface if one can be obtained from the other by cyclic permutations of the pairs in the continued fraction expansion of their sources. This remaining ambiguity is easy to take care of, so that we can now make a list of all closed geodesics corresponding to continued fraction expansions of a given period length. Moreover we know from eq. (38) that the length of the geodesic is a monotonically increasing function of the partial quotients, so it is straightforward to compute the number of primitive closed geodesics of a length not exceeding some chosen reasonable number. The result of such a calculation is given in fig. 4. Continuing this exercise on a computer one can see how eq. (29) emerges. (Curiously we were unable to find this calculation in the accessible literature, although it has been done before [22].)

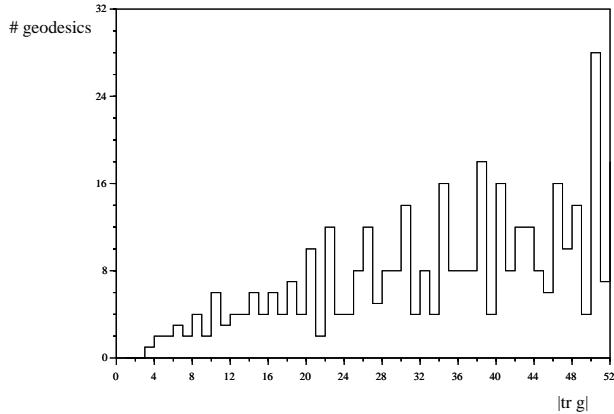


Figure 4: Degeneracies of the length spectrum: The first 52 levels.

The conclusion is that 2+1 gravity on the torus is a chaotic system according to the definition that we have adopted. Unlike the case of Bianchi models no approximation was involved. It may be felt that this chaos was introduced by sleight-of-hand since the system was in fact integrable before the modular group was declared to generate gauge symmetries. Indeed we are dealing with chaos of a very special kind, called "arithmetical chaos". Although the system is chaotic in the sense that the number of closed orbits not exceeding a given length grows exponentially, it is also very special because there are huge degeneracies in the length spectrum (caused by the fact that the number of possible lengths grows much more slowly). Closer investigation reveals [14] that in such situations the level statistics of the Laplace operator shows some features that resemble integrable systems much more than they resemble a generic chaotic system (in particular the level repulsion that is typical of the latter is missing here) so the feeling is justified to some extent.

5. A LOOSE END.

In the previous section we occupied ourselves with the action of the modular group on the Teichmüller space of Riemannian tori; the quotient space—the moduli space of Riemannian tori—is almost a smooth manifold since the modular group has only two elliptic conjugacy classes, and only the elliptic

members of the modular group have fixed points in \mathbf{H}^2 . The situation is dramatically different for the action of the modular group on the Teichmüller space of Lorentzian tori: Here every hyperbolic element of the modular group has fixed points inside the space, and we have already seen that there is an infinite number of inequivalent elements of this type.

The modular group is a subgroup of $PSL(2, \mathbf{R}) = SO(2, 1)$ and this is the isometry group of \mathbf{H}^2 and 1+1 dimensional de Sitter space alike. The action of the generators of the modular group is as follows. The generator T gives rise to a "null rotation" generated by a Killing vector that becomes null along the coordinate singularity that separates the two parts of de Sitter space in the description we gave above; its fixed points lie on the conformal boundary. In the half plane coordinates it is simply a translation in the x -direction. The generator S is a spatial rotation of de Sitter space; it has no fixed points and cannot be described in a single coordinate patch of the type used above. If we think of S as effecting an interchange of the basis elements in the dyad that defines the torus we see that this must be so whenever one of the elements is spacelike and the other timelike—the generator S will then transform a point representing a spacelike ξ_1 (say) into a point in the other coordinate patch where ξ_1 is timelike. Fig. 5 should be enough to make this clear.

Each hyperbolic element of the modular group has two fixed points inside de Sitter space, and two fixed points on each component of the conformal boundary separated from the fixed points in the interior by null lines that are left invariant by the transformation. The fixed points on the boundary are conjugate pairs of quadratic surds and conversely. This makes it easy to prove the next theorem:

Theorem 3: The action of the modular group on the Teichmüller space of Lorentzian tori is ergodic, in the sense that an arbitrary point can be transformed into an arbitrary coordinate neighbourhood of any other point.

Proof: We must show that there is a modular transformation taking an arbitrary point A into a given neighbourhood B_ϵ of another arbitrary point B . All neighbourhoods are regarded as coordinate neighbourhoods and we assume that the pair of points lies within some half plane coordinate patch. (There are exceptional pairs for which this fails, but they can easily be treated with an extension of the argument and will be ignored.)

We need to know that in any neighbourhood of any point there is a

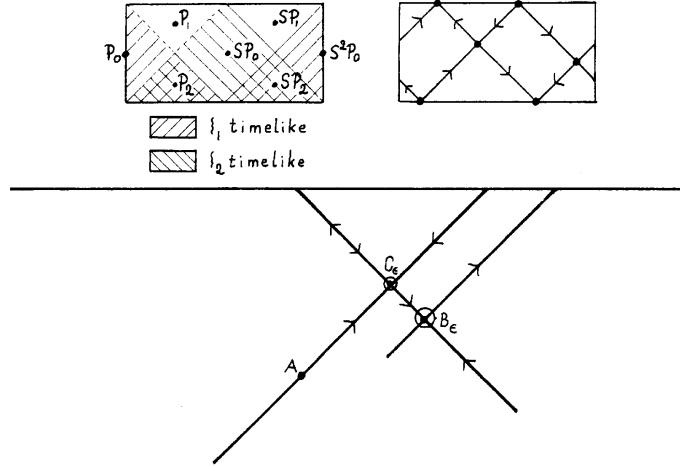


Figure 5: The action of S on 1+1 dimensional de Sitter space; also the null flow lines and the fixed points of a hyperbolic transformation; and a sketch of the proof (involving two different hyperbolic transformations) that the action of the modular group is ergodic.

hyperbolic modular transformation with a fixed point in that neighbourhood. This will be so if, given any two points x_{\pm} on the conformal boundary, we can find a conjugate pair of quadratic surds with one member arbitrarily close to each. But this is easy using the technology of the previous section. First modular transformations are used to show that it is enough to consider the case $x_+ > 1$, $-1 < x_- < 0$. Then one approximates x_+ and $-1/x_-$ with continued fractions to the desired accuracy. The sequence of integers that gives the continued fraction approximating $-1/x_-$ is then reversed and added to the sequence that approximates x_+ , and the resulting sequence is taken to be the period of a purely periodic continued fraction. Galois' theorem shows that we now have an approximation of x_+ whose conjugate surd approximates x_- . At the end we choose x_{\pm} to be null separated from the given point. They intersect at a fixed point, and we are done.

With this understanding, draw null lines through A and B meeting each other at the point C . Choose a suitable neighbourhood C_{ϵ} of C and a hyperbolic modular transformation with a fixed point in C_{ϵ} . Use this transformation to move the point A into C_{ϵ} . Then choose a hyperbolic modular transformation with a fixed point in B_{ϵ} and adjust the size of C_{ϵ} so that the

second transformation moves C_ϵ into B_ϵ .

Except for a speculative remark in the conclusions we have nothing to say about what this means.

6. OTHER SPACETIMES.

The final issue is to what extent the results described above are peculiar to flat spacetimes and to the genus one case. We confine our remarks to region I, where there are no closed timelike curves and the tori are spacelike. Consider first locally de Sitter spacetimes. 2+1 dimensional de Sitter spacetime can be described as the hypersurface

$$X^2 + Y^2 + Z^2 - U^2 = 1 \quad (41)$$

embedded in a four dimensional Minkowski space (with U as its time coordinate). Alternatively, it is the maximally symmetric vacuum solution to Einstein's equations with a positive cosmological constant λ . Again we choose two commuting and linearly independent Killing vectors

$$\xi_1 = \alpha J_{ZU} + \beta J_{XY} \quad \xi_2 = \gamma J_{ZU} + \delta J_{XY} . \quad (42)$$

They leave invariant the flat surfaces

$$U^2 - Z^2 = \sinh^2 \tau , \quad (43)$$

whose mean curvature is $K = 4 \cosh \tau$. Following the same steps as above we find that the invariant flat surfaces are turned into tori and that the evolution of the shape of these tori is given by a geodesic in Teichmüller space, with the interesting difference [15] that the evolution slows down and tends to a definite point in Teichmüller space as the parameter τ goes to infinity (while the area continues to grow). Explicitly

$$(x, y) = \frac{1}{\alpha^2 \tanh^2 \tau + \beta^2} (\alpha \gamma \tanh^2 \tau + \beta \delta, (\alpha \delta - \beta \gamma) \tanh \tau) \quad (44)$$

$$A = (\alpha\delta - \beta\gamma) \sinh \tau \cosh \tau . \quad (45)$$

The evolution stops because $\tanh \tau \rightarrow 1$ as $\tau \rightarrow \infty$. Note that this time the change of area as we move a distance L along the geodesic does depend on where we are on the geodesic. A subtlety should be mentioned also, namely that the universal covering space of the quotient spaces considered here is not, in general, de Sitter space itself but a "larger" incomplete spacetime of constant curvature [9] [23]; for the best explanation that we have to offer see ref. [24].

Why does the evolution stop in the interior of Teichmüller space? The answer is in fact obvious: In the de Sitter case future infinity \mathcal{J} is a spacelike surface transformed into itself by Γ . When we take the quotient we obtain an "asymptotic torus" with a definite conformal structure, and this is the endpoint of the geodesic in Teichmüller space. The area of this torus is not defined since \mathcal{J} is equipped with a conformal structure only. At this point the reader may object that \mathcal{J} is a sphere and that a discrete group like our Γ cannot act properly discontinuously on a sphere. This is true but irrelevant; in fact the covering space that we are using is not quite de Sitter space but an incomplete spacetime obtained by removing two timelike lines from de Sitter space, and afterwards going to the universal covering space. This means that \mathcal{J} is really a twice punctured sphere that has been "unrolled" to form a plane. This is explained in fig. 7 in ref. [24], where it can be seen that the invariant flat spacelike surfaces that were defined in the previous section do not encounter the timelike lines that were removed (except on \mathcal{J} itself).

For the genus one case then we find that the chaotic behaviour in the moduli space of tori is somehow "washed away" by the cosmological constant. It should however be noted that the flat torus universe is quite special in this regard. We can obtain locally flat spacetimes foliated by Riemann surfaces of higher genus by choosing Γ to be a discrete group—but this time not a free group—generated by non-commuting elements that in general are combinations of boosts and translations. These spacetimes are conformally static when Γ consists of pure boosts. As time passes the boost parts will dominate the translations and the solution will tend to a conformally static solution, that is to a definite point inside Teichmüller space. (This has been demonstrated with full rigour [25].) For the genus one case the evolution never stops for essentially the same reason; it is still true that eventually the

boost part of the generators will dominate but now this means that the shape of the torus degenerates so that we approach the boundary of Teichmüller space.

6. CONCLUSIONS.

The main new results of this paper are the explicit description of the moduli space of Lorentzian tori as the union of two half planes constituting a 1+1 dimensional de Sitter space, and the demonstration that the description of the 2+1 dimensional locally flat torus universe as a geodesic in Teichmüller space is valid on both sides of the Cauchy horizon. We also emphasized the analogy between these 2+1 dimensional spacetimes on the one hand, and mixmaster cosmology on the other. The difference between them is that the BKL approximation is exact in the former case. This is interesting because it shows that chaotic behaviour in general relativity should not in general be blamed on strong gravitational fields.

There are some open ends. We did not describe the extension to a geodesically complete spacetime, but this was mainly because it appears clear that this would give nothing new (compared to Misner's original work [16]). A more interesting open end is that the analogy to Bianchi IX cosmology holds only in the region where there are no closed timelike curves and the configuration space can be taken to be the moduli space of flat Riemannian tori, which is almost a smooth manifold. In the region with closed timelike curves we have to deal with the moduli space of Lorentzian tori, which is defined as the quotient of 1+1 dimensional de Sitter space by the modular group. But—as we demonstrated—the action of the modular group is now ergodic, so that the resulting quotient space is not easily described even as a set. It is our understanding that the desire to describe sets of this type is one of the main motivations behind non-commutative geometry [26]. It would be marvellous if one could follow this lead in such a way that an analogy with the singularity in 3+1 dimensional cosmologies could be drawn.

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